3.1

Equivalent Expressions

Decide whether each of the following is an equivalent expression to

$$x^4 - y^4$$

Explain your reasoning on each choice.

A. $(x^2)^2 - (y^2)^2$ Circle one: Yes No Can't determine

Explain your choice:

B. $(x^2 - y^2)(x^2 + y^2)$ Circle one: Yes No Can't determine

Explain your choice:

C. $(x + y)(x - y)(x^2 + y^2)$ Circle one: Yes No Can't determine

Explain your choice:

D. $(x^2 - y^2)^2$ Circle one: Yes No Can't determine

Explain your choice:

Teacher Notes: Equivalent Expressions

Questions to Consider About the Key Mathematical Concepts

Can students correctly identify equivalent expressions involving exponents? To what extent do they

- make sense of how expressions involving exponents can be rewritten to create equivalent expressions?
- reason whether certain expressions are equivalent to a given expression?
- describe the process of rewriting an expression to create equivalent ones using the properties of exponents and difference of squares?

Common Core Connection (CCSS.Math.Content.HSA-SSE.A.2)

Grade: High School

Domain: Algebra

Cluster:

Interpret the structure of expressions.

A2. Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Common Core Connection (CCSS.Math.Content.HSA-REI.B.4b)

Grade: High School

Domain: Algebra

Cluster:

Solve equations and inequalities in one variable.

B4b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation.

Uncovering Student Understanding About the Key Concepts

Using the Equivalent Expressions Probe can provide the following information about how the students are thinking about equivalent expressions involving exponents.

Do they		*Do they*
• apply the properties of exponents correctly?	OR	• inappropriately apply the properties of exponents?
• recognize when the difference of squares can be used to factor or rewrite an expression?	OR	• not realize that an expression can be factored or rewritten?
• recognize when expressions are or are not equivalent and why?	OR	• base their opinions on what "looks" equivalent?

Exploring Excerpts From Educational Resources and Related Research

Areas of consideration:

> The expression $(x + y)^2$ is often converted to $x^2 + y^2$ following the pattern of $(xy)^2 = x^2y^2$. Of course writing $(x + y)^2 = (x + y)(x + y)$ and using the distributive law (twice) helps clarify where the missing middle term $2xy$ comes from, in contrast to $(xy)^2$, which converts to x^2y^2 using only the commutative and associative laws. (National Council of Teachers of Mathematics [NCTM], 1999, p. 336)

> As stated in the Algebra Standard for grades 9 through 12, high school students are expected to "understand the meaning of equivalent forms of expressions, equations, inequalities, and relations" (p. 296). If that understanding is to occur, teachers must figure out how to open classroom discussions to questions about equivalence—to questions about why some methods preserve equivalence and others do not. (NCTM, 2003, p. 123)

> Many students have difficulty reading an expression. [They] cannot attach meaning to algebraic expressions. (Stepans, Schmidt, Welsh, Reins, & Saigo, 2005, p. 150)

> In general, if students engage extensively in symbolic manipulation before they develop a solid conceptual foundation for their work, they will be unable to do more than mechanical manipulation. The foundation for meaningful work with symbolic notation should be laid over a long time. (NCTM, 2000, p. 39)

Surveying the Prompts and Selected Responses in the Probe

The Probe consists of four related justified list items. The prompts and selected responses are designed to elicit understandings and common difficulties as described below:

If a student chooses	*It is likely that the student*
A. Yes, B. Yes, C. Yes, and D. No (correct responses)	• understands how to use the properties of exponents and the difference of squares to write equivalent expressions. ○ In example A, the terms x^4 and y^4 are rewritten as a power to a power $(x^2)^2$ and $(y^2)^2$. ○ In example B, the original expression is factored using the difference of squares. ○ Example C is factored one more time from example B. The difference of squares is used twice to completely factor the original expression. [See Sample Student Response 1] *Look for indication of the student's understanding in the written explanations of how the student got the answer.*
A. No or Can't determine B. No or Can't determine C. No or Can't determine	• lacks complete understanding of the properties of exponents and/or does not recognize factored forms of expressions. [See Sample Student Response 2]
D. Yes or Can't determine	• inappropriately applies a power into parentheses containing more than one term. Students are probably overgeneralizing the rule that allows $(x^2y^2)^2 = x^4y^4$. [See Sample Student Response 3]

Teaching Implications and Considerations

Ideas for eliciting more information from students about their understanding and difficulties:

- What does x^2 mean (x times x)? What does $(x^2)^2$ mean (x^2 times x^2)? How many xs are being multiplied in this expression (x times x times x times x)? How can four xs multiplied together be written?
- Expand $(xy)^2$ and $(x + y)^2$. What does each expression mean?
- Compare and contrast $(xy)^2$ and $(x + y)^2$.

- How is the difference of squares used with numbers? What happens when we use variables instead of numbers? In what way is the process the same/different?
- Can the difference of squares be used more than once in an expression?
- What does it mean to have an expression factored completely?

Ideas for planning instruction in response to what you learned from the results of administering the Probe:

- Provide opportunities for students to explore the properties of exponents with numbers before transferring the properties to variables. Investigate equivalent expressions with number bases before working with variable bases.
- With the use of technology, have students compare and contrast the graph of the original expression with the graphs of the other expressions. This can be a starting point for discussions on what makes each one equivalent or not.
- Have students expand (multiply) the expressions in examples B, C, and D to check for equivalency.

Sample Student Responses to Equivalent Expressions

Responses That Suggest Understanding

Sample Student Response 1

Probe Item A: Student chose Yes. Since x^4 is the same as $(x^2)^4$ and same for the y part, then this is equivalent. I checked with whole numbers to be sure.

Responses That Suggest Difficulty

Sample Student Response 2

Probe Item C: Student chose No. This has way too many terms to multiply to be just $x^4 - y^4$.

Sample Student Response 3

Probe Item D: Student chose Yes. Just distribute the 2 to each of the inside exponents.

Variation: Is It Simplified?

3.1Va

Circle the examples showing an appropriate use of "canceling digits" to simply an expression.

A.

$$\frac{\cancel{x}y}{\cancel{x}z} = \frac{y}{z}$$

B.

$$\frac{\cancel{m}+x}{\cancel{m}+n} = \frac{x}{n}$$

C.

$$\frac{\cancel{5}y}{\cancel{5}z} = \frac{y}{z}$$

D.

$$\frac{\cancel{c}d}{a\cancel{c}} = \frac{d}{a}$$

E.

$$\frac{3\cancel{m}+6n}{5\cancel{m}} = \frac{3+6n}{5}$$

F.

$$\frac{5\cancel{x}\cancel{y}z}{7\cancel{x}\cancel{y}} = \frac{z}{7}$$

Explain how you decide which one(s) to circle.

3.1Vb

Variation: Equivalent Expressions—Trig

Decide whether each of the following are equivalent expressions. Explain your reasoning on each choice.

A. Is $\sin 3x$ equivalent to $3 \sin x$?

Circle one: Yes No Can't determine

Explain your choice:

B. Is $\sin x$ equivalent to $\frac{1}{\csc x}$?

Circle one: Yes No Can't determine

Explain your choice:

C. Is $\sin(x + y)$ equivalent to $\sin x + \sin y$?

Circle one: Yes No Can't determine

Explain your choice:

D. Is $\sin \pi$ equivalent to $\sin 3\pi$?

Circle one: Yes No Can't determine

Explain your choice:

Variation: Equivalent Expressions—Logs

Decide whether each of the following are equivalent expressions. Explain your reasoning on each choice.

A. Is $\log x^3$ equivalent to $3 \log x$?

Circle one: Yes No Can't determine

Explain your choice:

B. Is $\log(x + y)$ equivalent to $\log x + \log y$?

Circle one: Yes No Can't determine

Explain your choice:

C. Is $\log xy$ equivalent to $x \log y$?

Circle one: Yes No Can't determine

Explain your choice:

D. Is $\log xy$ equivalent to $\log x + \log y$?

Circle one: Yes No Can't determine

Explain your choice: